Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 5, pp. 683–693, 2002

OPTIMUM SHAPES OF SPATIAL BODIES WITH A MINIMUM TOTAL RADIATIVE FLUX TO THE SURFACE

M. A. Arguchintseva[1] **and N. N. Pilyugin**[2]

UDC 533.6.11–535.23

A solution of a variational problem of a slender body with a minimum total radiative heat flux, moving in a gas with a constant velocity, is constructed. It is found that there are three types of the transverse contour of the optimum body: a circumference, a star-shaped contour, and a contour consisting of circle arcs and sectors of straight lines. The radiation parameter affects only the shape of the longitudinal contour and does not affect the optimum shape of the transverse contour. It is shown that the use of optimum spatial bodies allows a significant (more than 50%) decrease in the radiative heat flux to the body surface as compared to bodies of revolution with similar geometric characteristics.

Introduction. Spacecraft reentry with hypersonic velocities into the Earth atmosphere is accompanied by intense convective and radiative heating, which may lead to substantial thermochemical destruction of the surface [1]. At high Mach numbers (M $>$ 25–30) and Reynolds numbers, heat is transferred to the body surface mainly by radiation. One method of decreasing heat fluxes toward the body is the choice of the optimum shape of the reentry vehicle and its flight trajectory [2–4]. Most papers dealing with studying optimum bodies in terms of the heat flux considered problems of optimum longitudinal contours of axisymmetric and plane bodies [2]. Using the method of local variations, Arguchintseva and Pilyugin [4] found the optimum shapes of three-dimensional bodies with a given elliptic base cross section and a minimum total (convective and radiative) heating of the surface along the trajectory. It was shown that the total heat flux to the surface of a three-dimensional body is 30% lower than for an axisymmetric configuration under identical reentry conditions.

However, the issues of optimization of spatial configurations in terms of heat transfer are still poorly studied [3]. Therefore, it is of interest to consider the problems of finding optimum shapes of transverse contours of three-dimensional bodies from the viewpoint of minimization of the radiative heat flux.

1. Formulation of the Problem. We consider hypersonic motion of a three-dimensional body in a cylindrical coordinate system (r, θ, z) with the origin at the body apex and the z axis directed opposite the translational motion of the gas flow; θ is the angle formed by the radius r and the plane (x, z) of the Cartesian coordinate system (x, y, z).

We confine ourselves to consideration of a class of surfaces possessing homothetic properties. In this case, each cross section of the body perpendicular to the z axis should be geometrically similar to the base cross section. The surfaces of such bodes are described by the equation

$$f(r, \theta, z) = r - \varphi(z)R(\theta) = 0,$$

where the functions $\varphi(z)$ and $R(\theta)$ determine the longitudinal and transverse contours of the body, respectively.

We assume that the heat flux on the body surface depends on the local slope of the surface θ_w, i.e., on the angle between the external normal to the surface element and the free-stream velocity vector. Dependences of this kind are obtained by various methods for hypersonic flows around axisymmetric [5, 6] and plane [7] bodies. In the limiting case of a strongly radiating gas, the radiative flux to the spatial body is $q_R \sim \cos^3 \theta_w$ [8]. More generic power dependences of radiative heat fluxes to spatial bodies on $\cos \theta_w$ were considered in [1, 9] and were confirmed

[1]Irkutsk State University, Irkutsk 664003. [2]Institute of Mechanics of at the Moscow State University, Moscow 117192. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 5, pp. 55–68, September–October, 2002. Original article submitted October 8, 2001; revision submitted February 14, 2002.

by the numerical solution of problems of hypersonic flow around some spatial bodies [10, 11]. It was proved [9] that, if a three-dimensional body differs little from an axisymmetric one and the cross-sectional areas of these bodies are identical, then, in accordance with the "rule of areas," it is possible in some cases to recalculate heat fluxes to spatial bodies from the corresponding values of radiative fluxes to axisymmetric bodies [1].

Thus, we may assume that the dependence of the radiative heat flux on the local slope of the body surface is also valid for spatial bodies. As is noted above, this is confirmed in a number of limiting cases [1, 8, 9]. In addition, there are no other approximations of the distribution of radiative heat fluxes to spatial bodies in the currently available literature. In the present paper, the distribution of the local radiative heat flux q_R over the surface of a three-dimensional body was used in the first approximation in the form [1]

$$q_R = q_{R0}(-\langle \boldsymbol{n}, \boldsymbol{e}_z\rangle)^m, \qquad \langle \boldsymbol{n}, \boldsymbol{e}_z\rangle = -\dot\varphi(z)R(\theta)/[1 + (\dot R(\theta)/R(\theta))^2 + (\dot\varphi(z)R(\theta))^2]^{1/2},$$

where q_{R0} is the coefficient of radiative heat transfer at the critical point of the body, m is the radiation parameter determined by the emissivity of the medium and gas velocity ($m \in [3; 10]$) [10, 11], $\boldsymbol{n}$ is the unit vector of the external normal to the surface element, and $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$ are the unit vectors of the cylindrical coordinate system.

Note, similar dependences (with the corresponding values of the power m) are obtained in some other cases of heat transfer in a hypersonic flow around spatial bodies: for convective heat fluxes [12–14], in rarefied flows [15, 16], and also under the action of intense laser radiation on the body [17]. Therefore, the results of the present study can be used (as a first approximation) for other cases of spatial heat transfer.

Integrating the local radiative heat flux q_R along the side surface of the body S, we obtain the formula for the radiative heat-transfer coefficient

$$C_R = \frac{1}{L^2}\iint\limits_S \frac{q_R}{q_{R0}}\,ds, \quad ds = \frac{r}{\langle \boldsymbol{n}, \boldsymbol{e}_r\rangle}\,d\theta\,dz, \quad \langle \boldsymbol{n}, \boldsymbol{e}_r\rangle = \frac{1}{[1 + (\dot R(\theta)/R(\theta))^2 + (\dot\varphi(z)R(\theta))^2]^{1/2}},$$

where L is a prescribed body length along the z axis. In dimensionless variables [$\zeta = z/L$ and $\rho(\theta) = R(\theta)/L$], the formula for the radiative heat-transfer coefficient acquires the form

$$C_R = \int\limits_0^{2\pi}\int\limits_0^1 \frac{\varphi(\zeta)\dot\varphi^m(\zeta)\rho^{m+1}(\theta)\,d\theta\,d\zeta}{[1 + (\dot\rho(\theta)/\rho(\theta))^2 + (\dot\varphi(\zeta)\rho(\theta))^2]^{(m-1)/2}}.$$

In the class of slender bodies [$(\dot\varphi(\zeta)\rho(\theta))^2 \ll 1$], the expression for C_R can be represented as the product of the functionals $J_1(\varphi)$ and $J_2(\rho)$ depending on the shape of the longitudinal [$\varphi(\zeta)$] or transverse [$\rho(\theta)$] contours of the body, respectively:

$$C_R = J_1(\varphi)J_2(\rho), \quad J_1(\varphi) = \int\limits_0^1 \varphi(\zeta)\dot\varphi^m(\zeta)\,d\zeta, \quad J_2(\rho) = \int\limits_0^{2\pi} \frac{\rho^{2m}(\theta)\,d\theta}{[\rho^2(\theta) + \dot\rho^2(\theta)]^{(m-1)/2}}. \tag{1.1}$$

Thus, the initial problem of finding the shape of a three-dimensional body, which minimizes the radiative heating of the surface, can be divided into two subproblems: 1) on the optimum longitudinal contour of the body; 2) optimum transverse contour of the body.

As restrictions imposed onto the body surface, we consider the following isoperimetric conditions:

— on the given volume of the body V,

$$\int\limits_0^1 \varphi^2(\zeta)\,d\zeta \int\limits_0^{2\pi} \rho^2(\theta)\,d\theta = \frac{2V}{L^3}; \tag{1.2}$$

— on the area S of the wetted body surface,

$$\int\limits_0^{2\pi}\int\limits_0^1 \varphi(\zeta)\rho(\theta)[1 + (\dot\rho(\theta)/\rho(\theta))^2 + (\dot\varphi(\zeta)\rho(\theta))^2]^{1/2}\,d\theta\,d\zeta = \frac{S}{L^2}$$

or, in the class of slender bodies, this restriction takes the form

$$\int\limits_0^1 \varphi(\zeta)\,d\zeta \int\limits_0^{2\pi} [\rho^2(\theta) + \dot\rho^2(\theta)]^{1/2}\,d\theta = \frac{S}{L^2}; \tag{1.3}$$

— on the area of the base cross section of the body S_b,

$$\int_0^{2\pi} \rho^2(\theta)\, d\theta = \frac{2S_b}{L^2}; \tag{1.4}$$

— on the perimeter of the base cross section P,

$$\int_0^{2\pi} \sqrt{\rho^2(\theta) + \dot\rho^2(\theta)}\, d\theta = \frac{P}{L}. \tag{1.5}$$

We also consider an additional restriction on the limiting value of the wave drag of the body. The wave-drag functional D has the form [18]

$$\frac{D}{q} = \iint_S (-\langle \boldsymbol{n}, \boldsymbol{e}_z\rangle)^3\, ds, \qquad ds = \frac{r}{\langle \boldsymbol{n}, \boldsymbol{e}_r\rangle}\, d\theta\, dz,$$

where q is the dynamic pressure. Then, in the class of slender bodies, the restriction on the body drag yields the inequality

$$\int_0^1 \varphi(\zeta)\dot\varphi^3(\zeta)\, d\zeta \int_0^{2\pi} \frac{\rho^6(\theta)\, d\theta}{\dot\rho^2(\theta) + \rho^2(\theta)} \leqslant \frac{D_*}{qL^2}, \tag{1.6}$$

where D_* is a prescribed limiting value of the wave drag.

2. Optimum Longitudinal Contour. We consider the following optimization problem: among smooth functions $\varphi(\zeta)$ that describe the longitudinal contour of the body and satisfy the boundary conditions $\varphi(0) = 0$ and $\varphi(1) = 1$, we have to find a function that minimizes the heat-flux integral $J_1(\varphi)$ (1.1).

It should be noted that the problems of finding optimum longitudinal contours of bodies from the viewpoint of minimization of radiative heating of the surface are considered in many papers (see the review in [2]). Therefore, we mention only the main aspects of solving the optimization problem posed.

The function $\varphi(\zeta)$ minimizing the functional $J_1(\varphi)$ should satisfy the Euler equation

$$(1 - m)\varphi(\zeta)\dot\varphi^m(\zeta) = C \qquad (C = \text{const}),$$

which, with allowance for the boundary conditions, has the solution

$$\varphi(\zeta) = \zeta^{m/(m+1)}. \tag{2.1}$$

The resultant curve satisfies the Legendre condition. The value of the functional at the extreme line is determined by the formula $J_1^{\min}(\varphi) = [m/(m+1)]^m$.

Taking into account (2.1), we obtain the following isoperimetric conditions on the body volume V (1.2) and on the area of the wetted surface S (1.3), as well as the restriction on the wave drag of the body (1.6):

$$\int_0^{2\pi} \rho^2(\theta)\, d\theta = V_*, \qquad V_* = \frac{V}{L^3}\,\frac{2(3m+1)}{m+1} = \text{const}; \tag{2.2}$$

$$\int_0^{2\pi} \sqrt{\rho^2(\theta) + \dot\rho^2(\theta)}\, d\theta = S_*, \qquad S_* = \frac{S}{L^2}\,\frac{2m+1}{m+1} = \text{const}; \tag{2.3}$$

$$\int_0^{2\pi} \frac{\rho^6(\theta)\, d\theta}{\dot\rho^2(\theta) + \rho^2(\theta)} \leqslant C_{D_*}, \qquad C_{D_*} = \frac{D_*}{qL^2}\,\frac{(m+1)^2(m-1)}{m^3} = \text{const}. \tag{2.4}$$

3. Optimum Transverse Contour. The problem of the optimum transverse contour of the body is formulated as follows. Among piecewise-smooth functions $\rho(\theta)$ that satisfy the condition of the closed contour $\rho(0) = \rho(2\pi)$, isoperimetric conditions on the volume V_* (2.2) and on the area of the wetted body surface S_* (2.3), and also the restriction on the limiting admissible value of the wave drag C_{D_*} (2.4), we have to find a function that minimizes the integral of the radiative heat flux $J_2(\rho)$ (1.1).

To study the problem posed, we use the approach suggested in [18]. We write the Lagrangian functional

$$J(\rho) = \int\limits_0^{2\pi} F(\lambda_1, \lambda_2, \lambda_3, \rho, \dot\rho)\, d\theta, \tag{3.1}$$

where $F(\lambda_1, \lambda_2, \lambda_3, \rho, \dot\rho) = \rho^{2m}(\theta)/[\rho^2(\theta) + \dot\rho^2(\theta)]^{(m-1)/2} + \lambda_1 \rho^2(\theta) + \lambda_2 \sqrt{\rho^2(\theta) + \dot\rho^2(\theta)} + \lambda_3 \rho^6(\theta)/(\rho^2(\theta) + \dot\rho^2(\theta))$ (λ_1, λ_2, and λ_3 are indeterminate constant Lagrangian multipliers).

It should be noted that the structure of the objective functional (3.1) remains unchanged if, instead of the isoperimetric conditions on V_* (2.2) and S_* (2.3), we consider isoperimetric conditions on the area of the base cross section of the body S_b (1.4) and its perimeter P (1.5), respectively. It follows from Eqs. (1.4), (1.5), (2.2), (2.3) that slender bodies with an optimum longitudinal contour (2.1) have the following relations of their geometric characteristics:

$$S_* = P/L, \qquad S_b = V_* L^2/2. \tag{3.2}$$

The function that is the solution of the initial variational problem should satisfy the following conditions:

— Euler equation for the Lagrangian functional (3.1)

$$F - \dot\rho F_{\dot\rho} = \frac{\rho^{2m}(\theta)[\rho^2(\theta) + m\dot\rho^2(\theta)]}{[\rho^2(\theta) + \dot\rho^2(\theta)]^{(m+1)/2}} + \lambda_1 \rho^2(\theta) + \frac{\lambda_2 \rho^2(\theta)}{\sqrt{\rho^2(\theta) + \dot\rho^2(\theta)}}$$

$$+ \lambda_3 \frac{\rho^6(\theta)[\rho^2(\theta) + 3\dot\rho^2(\theta)]}{[\rho^2(\theta) + \dot\rho^2(\theta)]^2} = C = \text{const}; \tag{3.3}$$

— condition of transversality that agrees with the end conditions

$$F_{\dot\rho}\Big|_0 = F_{\dot\rho}\Big|_{2\pi} = 0,$$

$$F_{\dot\rho} = \dot\rho(\theta)\Big(\frac{(1-m)\rho^{2m}(\theta)}{[\rho^2(\theta) + \dot\rho^2(\theta)]^{(m+1)/2}} + \frac{\lambda_2}{\sqrt{\rho^2(\theta) + \dot\rho^2(\theta)}} - \lambda_3 \frac{2\rho^6(\theta)}{[\rho^2(\theta) + \dot\rho^2(\theta)]^2}\Big); \tag{3.4}$$

— necessary Legendre condition

$$F_{\dot\rho\dot\rho} = \frac{(m-1)\rho^{2m}(\theta)}{[\rho^2(\theta) + \dot\rho^2(\theta)]^{(m+3)/2}} \Big(m\dot\rho^2(\theta) - \rho^2(\theta) + \frac{\lambda_2}{m-1} \frac{[\rho^2(\theta) + \dot\rho^2(\theta)]^{m/2}}{\rho^{2m-2}(\theta)}$$

$$+ \frac{2\lambda_3}{m-1} \frac{[\rho^2(\theta) + \dot\rho^2(\theta)]^{(m-3)/2}}{\rho^{2(m-3)}(\theta)} (3\dot\rho^2(\theta) - \rho^2(\theta))\Big) \geqslant 0; \tag{3.5}$$

— condition of supplementary slackness

$$\lambda_3 \Big(\int\limits_0^{2\pi} \frac{\rho^6(\theta)\, d\theta}{\dot\rho^2(\theta) + \rho^2(\theta)} - C_{D_*} \Big) = 0; \tag{3.6}$$

— Weierstrass–Erdmann conditions at corner points, which may be of two types: some points have arbitrary positions, and other points lie on the circumferences $\rho = \rho_0$ and $\rho = \rho_f$ (ρ_0 and ρ_f are the end radii of the transverse contour). In the first case, we obtain the relations

$$\Delta[F - \dot\rho F_{\dot\rho}] = 0, \qquad \Delta[F_{\dot\rho}] = 0, \tag{3.7}$$

where $\Delta[\,\cdot\,]$ is the difference in values calculated on the right (plus sign) and on the left (minus sign) of the corner point θ_c. Conditions (3.7) are satisfied for $\dot\rho_+(\theta_c) = \pm\infty$, $\dot\rho_-(\theta_c) = \mp\infty$, and $\lambda_2 = 0$. In the second case, only the first condition of (3.7) remains, which reduces to the relation $\dot\rho_+(\theta_c) + \dot\rho_-(\theta_c) = 0$. In addition, it follows from conditions (3.7) that the value of the constant C remains the same on all arcs of the extreme line.

Hence, any two neighboring arcs that form the extreme-line cycle are symmetric with respect to the ray passing through the corner point. Then the optimum transverse contour may consist of an integer number n of symmetric cycles, each occupying the angle $2\pi/n$. In this case, the condition of the closed transverse contour is satisfied. In analyzing the shape of the extreme line consisting of symmetric cycles, it is sufficient to study an arc that forms a half of the cycle.

With allowance for the above said, the functional of the problem and the restrictions on the body shape take the following form:

$$J_2(\rho, n) = 2n \int_0^{\pi/n} \frac{\rho^{2m}(\theta)\, d\theta}{[\rho^2(\theta) + \dot{\rho}^2(\theta)]^{(m-1)/2}} \to \min; \tag{3.8}$$

$$2n \int_0^{\pi/n} \rho^2(\theta)\, d\theta = V_*; \tag{3.9}$$

$$2n \int_0^{\pi/n} \sqrt{\rho^2(\theta) + \dot{\rho}^2(\theta)}\, d\theta = S_*; \tag{3.10}$$

$$2n \int_0^{\pi/n} \frac{\rho^6(\theta)\, d\theta}{\dot{\rho}^2(\theta) + \rho^2(\theta)} \leqslant C_{D_*}. \tag{3.11}$$

The radii at the end points $\theta_0 = 0$ and $\theta_f = \pi/n$ are assumed to be free. The Lagrangian functional for problem (3.8)–(3.11) is

$$J(\rho, n) = 2n \int_0^{\pi/n} F(\lambda_1, \lambda_2, \lambda_3, \rho, \dot{\rho})\, d\theta,$$

where the integrand $F(\lambda_1, \lambda_2, \lambda_3, \rho, \dot{\rho})$ is determined by Eq. (3.1).

We consider the necessary conditions of the extremum of the functional $J(\rho, n)$. The conditions to be satisfied on the extreme-line arc are the first integral of the Euler equation (3.3), which is studied together with the conditions of transversality (3.4) at the end points $\theta_0 = 0$ and $\theta_f = \pi/n$ and the condition of supplementing slackness (3.6).

The optimizing condition on the number of extreme-line arcs reduces to vanishing of the derivative of the integral $J(\rho, n)$ with respect to the parameter n [18] (n is assumed to be real):

$$\int_0^{\pi/n} F\, d\theta - \frac{\pi}{n}\, [F]_{\pi/n} = 0. \tag{3.12}$$

Among the solutions of Eq. (3.12), only the integer values of the parameter n that have a physical meaning are chosen. In the general case, the use of this approach can lead to the loss of possible solutions or to the absence of integer solutions of Eq. (3.12). However, as is shown below, the objective functional $J(\rho, n)$ has no explicit dependence on the parameter n. Hence, we have $dJ/dn = 0$ for all (including integer) values of n. Note that the use of the additional condition (3.12) allows obtaining analytical solutions of the initial variational problem, which satisfy both the necessary conditions of the extremum of the functional (3.3)–(3.7) and restriction (3.12).

Taking into account (3.4) and introducing the notation $\Phi = \dot{\rho} F_{\dot{\rho}}$, we integrate the Euler equation (3.3) on the interval $[0, \pi/n]$:

$$\int_0^{\pi/n} F\, d\theta - \int_0^{\pi/n} \Phi\, d\theta = \int_0^{\pi/n} C\, d\theta \equiv \frac{\pi}{n}\, [F]_{\pi/n}. \tag{3.13}$$

From Eqs. (3.12) and (3.13) and the condition of transversality (3.4), it follows that the optimum transverse contour should satisfy the conditions

$$\int_0^{\pi/n} \Phi\, d\theta = 0, \qquad \Phi\Big|_0 = \Phi\Big|_{\pi/n} = 0. \tag{3.14}$$

The following cases should be considered:

(a) the function Φ is identically equal to zero at all points of the segment $[0, \pi/n]$;

(b) the function Φ changes its sign at the segment $[0, \pi/n]$ and, hence, equals zero not only at the ends $\theta_0 = 0$, $\theta_f = \pi/n$ but also in intermediate points.

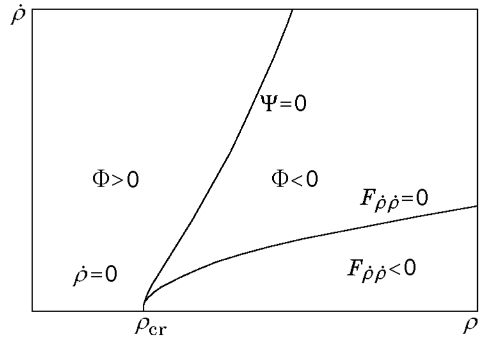

Fig. 1. Behavior of the function Φ on the plane $(\rho, \dot\rho)$.

In case (a), we obtain the relations

$$\dot\rho(\theta) = 0; \tag{3.15}$$

$$\frac{(1-m)\rho^{2m}(\theta)}{[\rho^2(\theta)+\dot\rho^2(\theta)]^{(m+1)/2}} + \frac{\lambda_2}{\sqrt{\rho^2(\theta)+\dot\rho^2(\theta)}} - \frac{2\lambda_3\rho^6(\theta)}{[\rho^2(\theta)+\dot\rho^2(\theta)]^2} = 0. \tag{3.16}$$

If the restriction on the wave drag (3.11) on the extreme line is fulfilled as a rigorous inequality, the condition of supplementing slackness (3.6) yields $\lambda_3 = 0$ $(i = 1)$. If $\lambda_3 \neq 0$ and $\lambda_2 = 0$ $(i = 2)$, the problem without the restriction on the area of the body surface is considered, where the condition of the wave drag (3.11) is fulfilled as a rigorous equality. For the cases considered, Eq. (3.16) reduces to the form

$$\Psi = \dot\rho^2(\theta) + \rho^2(\theta) - a_i^2\rho^4(\theta) = 0, \qquad i = 1, 2,$$
$$a_1 = [(m-1)/\lambda_2]^{1/m}, \qquad a_2 = [(1-m)/(2\lambda_3)]^{1/(m-3)}. \tag{3.17}$$

Considering the function Φ on the plane $(\rho, \dot\rho)$ (Fig. 1), we can draw the following conclusions. Dependences (3.15) and (3.17) satisfy the Legendre condition (3.5) for $\rho \leqslant \rho_{\rm cr}$ ($\rho_{\rm cr} = 1/a_i$ and $i = 1, 2$) and $\rho \geqslant \rho_{\rm cr}$, respectively. In this case, the function Φ is positive on the left of the line $\Psi = 0$ and negative on the right of it. Assuming that the end points of the extreme line ($\theta_0 = 0$ and $\theta_f = \pi/n$) are located on the line $\Phi = 0$ in an arbitrary manner, we have to study three classes of bodies.

4. Transverse Contour of Bodies of Class I. Let the end points of the extreme line lie on the line (3.15) $[\dot\rho(0) = \dot\rho(\pi/n) = 0]$. Since the solution of the differential equation (3.15) has the form

$$\rho(\theta) = C_1, \tag{4.1}$$

the optimum body of class I is a body of revolution. The constant C_1 is determined from the prescribed isoperimetric conditions: $C_1 = \sqrt{V_*/(2\pi)}$ in the case of a prescribed volume of the body (3.9) and $C_1 = S_*/(2\pi)$ in the case of a prescribed area of the side surface of the body (3.10). For the isoperimetric conditions (3.9) and (3.10) to be satisfied simultaneously, the input parameters of the problem should be related as $V_* = S_*^2/(2\pi)$.

The integral of the radiative heat flux for the transverse contour (4.1) is $J_2^{\rm min} = 2\pi C_1^{m+1}$.

5. Transverse Contour of Bodies of Class II. We consider a class of bodies where both end points lie on the curve $\Psi = 0$ ($\Psi|_0 = 0$ and $\Psi|_{\pi/n} = 0$). In this case, the transverse contour of the body satisfies the differential equation (3.17) whose solution is

$$\rho(\theta) = 1/(a_i\cos(\theta + C_2)) \qquad (C_2 = \text{const}, \quad i = 1, 2). \tag{5.1}$$

Expression (5.1) is the equation of a straight line, which is the side of a star-shaped transverse contour of the body. We find the relation of the unknown parameters a_i and C_2 with the prescribed geometric characteristics of the body.

1. We study the case $i = 1$ where the volume V_* ($\lambda_1 \neq 0$) and the area of the wetted surface of the body S_* ($\lambda_2 \neq 0$) are known, and the condition on the limiting wave drag of the body (3.11) is fulfilled as a rigorous inequality ($\lambda_3 = 0$). From the isoperimetric conditions (3.9) and (3.10), we obtain

$$a_1 = S_*/V_*, \qquad S_*^2/V_* = 2n(\tan(\pi/n + C_2) - \tan C_2). \tag{5.2}$$

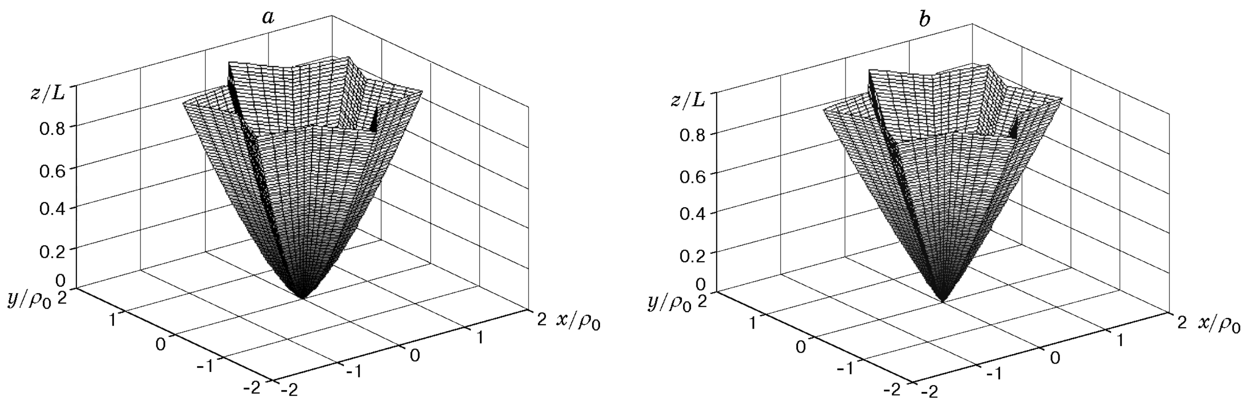

Fig. 2. Optimum shapes of bodies of class II [$n = 6$ and $S_*^2/V_* = 3n\tan(\pi/n)$] for $m = 3$ (a) and 10 (b).

The optimizing condition (3.12) for the number n of extreme-line arcs reduces to

$$\left[m\left(\frac{\lambda_2}{m-1}\right)^{(m-1)/m} + \lambda_1\right]\left\{\int\limits_0^{\pi/n} \rho^2(\theta)\,d\theta - \frac{\pi}{n}\,\rho^2\left(\frac{\pi}{n}\right)\right\} = 0.$$

The expression in braces cannot vanish, since thereby the function $\rho(\theta)$ does not satisfy the Euler equation (3.3). Hence, we have the following relation between the Lagrangian multipliers:

$$\lambda_1 = -m(\lambda_2/(m-1))^{(m-1)/m}, \qquad \lambda_2 = (m-1)(V_*/S_*)^m. \tag{5.3}$$

Then, condition (3.12) is satisfied for an arbitrary number of cycles n, and the integral for the radiative heat flux (3.8) is independent of n:

$$J_2^{\min} = S_*^{1-m} V_*^m. \tag{5.4}$$

Hence, the radiative heating of the surface is identical for optimum star-shaped bodies with different number of rays but identical values of the parameters m, S_*, and V_*.

By means of simple substitution, it can be shown that the obtained solutions (5.1)–(5.3) satisfy the initial Euler equation (3.3), condition of transversality (3.4), and Legendre condition (3.5). It follows from condition (3.5) that bodies of class II are obtained for $S_*^2/V_* \geqslant 2n\tan(\pi/n)$. In the case of a rigorous equality, the equation of the body shape reduces to the relation

$$\rho(\theta) = 1/(a_i \cos\theta) \qquad (C_2 = 0), \tag{5.5}$$

which is the side of a regular polygon described around the circumference $\rho = 1/a_i$. As $n \to \infty$, we have $S_*^2/V_* \to 2\pi$, and the optimum transverse contour is a circumference.

2. We consider the case $i = 2$, where the body volume V_* ($\lambda_1 \neq 0$) and the restriction on the wave drag of the body ($\lambda_3 \neq 0$) are prescribed, and the area of the body surface S_* is free ($\lambda_2 = 0$). Then, the isoperimetric conditions (3.9) and (3.11) yield

$$a_2 = \sqrt{V_*/C_{D_*}}\,, \qquad V_*^2/C_{D_*} = 2n(\tan(\pi/n + C_2) - \tan C_2).$$

The optimizing condition on the number of cycles of the extreme line (3.12) is satisfied for all n for

$$\lambda_1 = \frac{m-3}{2}\left(\frac{C_{D*}}{V_*}\right)^{(m-1)/2}, \qquad \lambda_3 = \frac{1-m}{2}\left(\frac{C_{D*}}{V_*}\right)^{(m-3)/2}. \tag{5.6}$$

Hence, the minimum value of the functional of the radiative flux on the extreme line is independent of the number of cycles and is calculated by the formula

$$J_2^{\min} = V_*^{(3-m)/2} C_{D_*}^{(m-1)/2}. \tag{5.7}$$

Bodies of class II with prescribed V_* and C_{D_*} are obtained with the following restriction on the input parameters of the problem: $V_*^2/C_{D_*} \geqslant 2n\tan(\pi/n)$.

Based on the resultant analytical expressions, we calculated the optimum shapes of bodies of class II in a wide range of input parameters. Figure 2 shows the star-shaped configurations with restrictions on the geometric

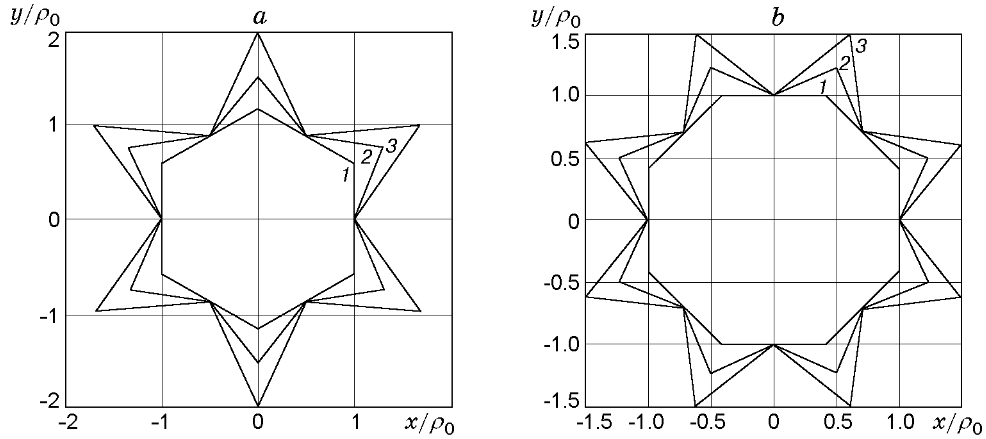

Fig. 3. Dynamics of the transverse contour of bodies of class II for $n = 6$ (a) and 8 (b) and $S_*^2/V_* = 2n\tan(\pi/n)$ (contour 1), $3n\tan(\pi/n)$ (contour 2), and $5n\tan(\pi/n)$ (contour 3).

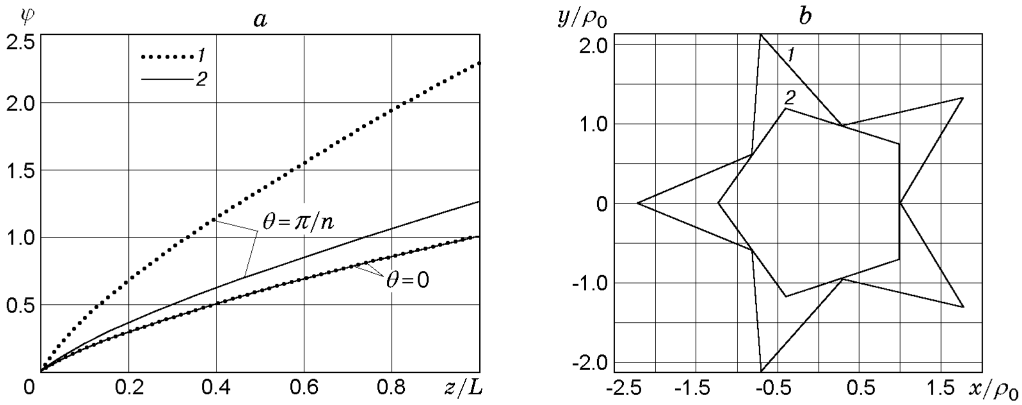

Fig. 4. Longitudinal (a) and transverse (b) contours of bodies of class II ($n = 5$ and $m = 5$) for $V_*^2/C_{D_*} = 5n\tan(\pi/n)$ (contour 1) and $2n\tan(\pi/n)$ (contour 2).

characteristics S_* and V_* for $n = 6$, $m = 3$ and 10, and $S_*^2/V_* = 3n\tan(\pi/n)$. The radiation parameter m affects only the longitudinal contour of the body; the shape of the transverse contour is independent of m.

The dynamics of the transverse contour of the body for $n = 6$ and 8 and various values of S_*^2/V_* is shown in Fig. 3. It is seen from Fig. 3 that, for $S_*^2/V_* \to 2n\tan(\pi/n)$, the ratio of the end radii ρ_f/ρ_0 decreases, and the cross-sectional shape of the body tends to a regular polygon.

Figure 4 shows the optimum longitudinal (a) and transverse (b) contours of star-shaped configurations in the case of prescribed V_* and C_{D_*} for $n = 5$, $m = 5$, and $V_*^2/C_{D_*} = 2n\tan(\pi/n)$ and $5n\tan(\pi/n)$.

6. Transverse Contour of Bodies of Class III. We consider a class of bodies where the initial point of the contour lies on the line (3.15) $\dot\rho|_0 = 0$ and the end point lies on the line (3.17) $\Psi|_{\pi/n} = 0$. Thus, the contour of class III contains an arc of the circumference and a straight line tangent to the circumference. It follows from Fig. 1 that the point of transformation of the circumference arc to the straight line corresponds to $\rho_0 = \rho_{\rm cr} \equiv 1/a_i$ ($i = 1, 2$) due to the continuity of the contour. Then, the extreme line can be represented analytically as

$$\rho = \begin{cases} 1/a_i, & 0 \leqslant \theta \leqslant \varepsilon_i, \\ 1/[a_i\cos(\theta - \varepsilon_i)], & \varepsilon_i \leqslant \theta \leqslant \pi/n, \end{cases} \qquad (6.1)$$

where ε_i ($i = 1, 2$) is the angle of junction of the extreme-line arcs, which is determined from the corresponding isoperimetric conditions: $S_*^2/V_* = 2n[\varepsilon_1 + \tan(\pi/n - \varepsilon_1)]$ and $a_1 = S_*/V_*$ in the case of prescribed V_* (3.9) and S_* (3.10); $V_*^2/C_{D_*} = 2n[\varepsilon_2 + \tan(\pi/n - \varepsilon_2)]$ and $a_2 = \sqrt{V_*/C_{D_*}}$ in the case of prescribed V_* (3.9) and C_{D_*} (3.11).

Note, from condition (3.12) on the number of cycles n, it follows that the optimum solutions of Eq. (6.1) are valid only if the corresponding restrictions (5.3) or (5.6) on the Lagrangian multipliers are satisfied. Thus, the resultant optimum shapes of bodies of class III satisfy the Euler equation (3.3), the condition of transversality (3.4),

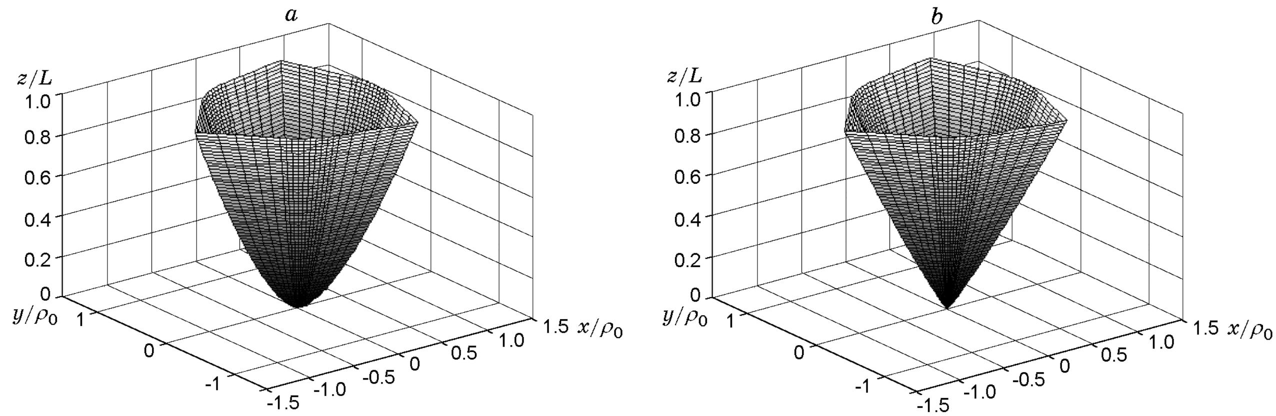

Fig. 5. Optimum shapes of bodies of class III [$n = 3$ and $S_*^2/V_* = 1.5n\tan(\pi/n)$] for $m = 4$ (a) and 10 (b).

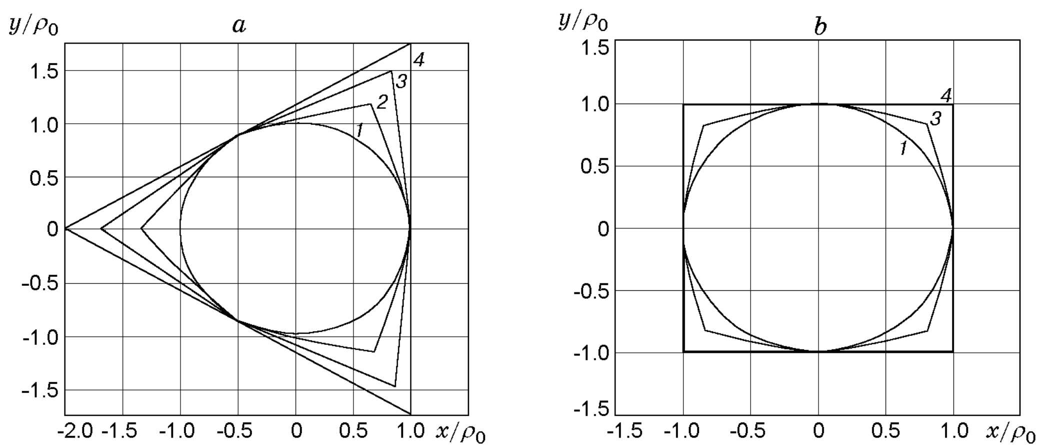

Fig. 6. Dynamics of the transverse contour of bodies of class III for $n = 3$ (a) and 4 (b) and $V_*^2/C_{D_*} = 2\pi$ (contour 1), $1.4n\tan(\pi/n)$ (contour 2), $1.7n\tan(\pi/n)$ (contour 3), and $2n\tan(\pi/n)$ (contour 4).

and the Legendre condition (3.5), as well as the optimizing condition (3.12).

The functional of the radiative heat flux for the extreme line (6.1) is determined by formulas (5.4) or (5.7) and does not depend on the number of cycles.

Figure 5 shows the optimum shapes of bodies of class III with prescribed parameters S_* and V_* for $n = 3$, $m = 4$ and 10, and $S_*^2/V_* = 1.5n\tan(\pi/n)$. The dynamics of the transverse contour of the body for $n = 3$ and 4 and various values of V_*^2/C_{D_*} is shown in Fig. 6.

It follows from Figs. 5 and 6 that, for S_*^2/V_* and V_*^2/C_{D_*} tending to $2n\tan(\pi/n)$ ($\varepsilon_i = 0$), the limiting case of the optimum transverse contour of class III is a regular polygon (5.5). The other limiting case, for S_*^2/V_* and V_*^2/C_{D_*} tending to 2π ($\varepsilon_i = \pi/n$), is a circumference of radius $\rho_0 = 1/a_i$.

7. Discussion of Results. The solutions obtained in Secs. 4–6 correspond to case (a) (see Sec. 3) where the function Φ is identically equal to zero on the segment $[0;\pi/n]$. We study case (b) where the function Φ can change its sign on the segment $[0;\pi/n]$ and, hence, equals zero not only at the end of the segment but also at a finite number of intermediate points.

We consider the above-mentioned classes of bodies:

— for bodies of class I ($\dot\rho|_0 = \dot\rho|_{\pi/n} = 0$), the function is $\Phi \equiv 0$ (see Fig. 1) or $\Phi > 0$, which contradicts the integral condition (3.14);

— for bodies of class II ($\Psi|_0 = \Psi|_{\pi/n} = 0$) and class III ($\dot\rho|_0 = \Psi|_{\pi/n} = 0$), the function Φ cannot change its sign on the segment $[0;\pi/n]$, since this contradicts the fact that the extreme line has an infinite number of intersections with the line $\Psi = 0$.

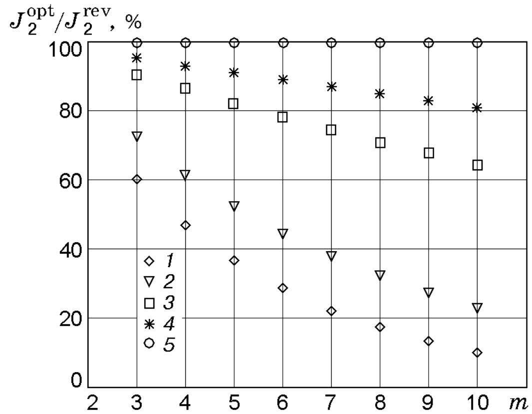

Fig. 7. Dependence $J_2^{\rm opt}/J_2^{\rm rev}$ on the radiation parameter m for $S_*^2/V_* = 10.3923$ (1), 8.6603 (2), 6.9282 (3), 6.5818 (4), and 2π (5).

Indeed, the first integral of the Euler equation (3.3) and Eq. (3.17) should be satisfied simultaneously at all points of intersection. Then, for prescribed S_*, V_* and S_*, C_{D_*}, the following relations are valid, respectively:

$$\Big[m\Big(\frac{\lambda_2}{m-1}\Big)^{(m-1)/m}+\lambda_1\Big]\rho^2(\theta)=C, \qquad \Big[\frac{3-m}{2}\Big(\frac{2\lambda_3}{1-m}\Big)^{(m-1)/(m-3)}+\lambda_1\Big]\rho^2(\theta)=C. \tag{7.1}$$

Since the restrictions on the Lagrangian multipliers (5.3), (5.6) are satisfied for bodies of classes II and III, the factors in square brackets in (7.1) and the constant C (3.3) vanish. Hence, Eqs. (7.1) are satisfied for an arbitrary value of ρ, and the extreme line has an infinite number of intersection points with the curve $\Psi = 0$. The proved statement justifies the validity of the solutions obtained.

Thus, bodies possessing optimum transverse contours from the viewpoint of minimization of radiative heating can be of four types:

1) bodies of class I (4.1) with circular cross sections ($\rho \leqslant \rho_{\rm cr}$);

2) bodies of class II (5.1) with star-shaped cross sections [$\rho \geqslant \rho_{\rm cr}$, $S_*^2/V_* \geqslant 2n\tan(\pi/n)$, and $V_*^2/C_{D_*} \geqslant 2n\tan(\pi/n)$];

3) bodies of class III (6.1) with star-shaped cross sections and circular inclusions ($\rho \geqslant \rho_{\rm cr}$, $S_*^2/V_* \in [2\pi; 2n\tan(\pi/n)]$, and $V_*^2/C_{D_*} \in [2\pi;\, 2n\tan(\pi/n)]$);

4) pyramidal bodies with cross sections in the form of regular polygons (5.5), which are subsets of bodies of classes II and III for $S_*^2/V_* = 2n\tan(\pi/n)$ and $V_*^2/C_{D_*} = 2n\tan(\pi/n)$.

The coefficients of radiative heating of the transverse contours of bodies of classes II and III depend on the radiation parameter of the gas m and specified geometric characteristics of the body but do not depend on the number of extreme-line cycles n. Thus, for bodies of classes II and III possessing identical prescribed parameters m, S_*, and V_* or m, C_{D_*}, and V_*, radiative heating of the surface is identical, and the number of rays is smaller for bodies of class III. The radiation parameter m affects only the shape of the longitudinal contour of the body $\varphi(\zeta)$ (2.1) and has no effect on the optimum shape of the transverse contour $\rho(\theta)$.

The ratios of the coefficients of radiative heating of transverse contours of optimum bodies of classes II and III $J_2^{\rm opt}$ and equivalent bodies of revolution $J_2^{\rm rev}$ are plotted in Fig. 7 as functions of the radiation parameter m for various values of S_*^2/V_*. An equivalent body of revolution is understood as a body of revolution that has the same length L ($L = 1$) and volume V_* as an optimum body. For example, for $n = 4$, curves 1 and 2 correspond to bodies of class II and curves 3–5 to bodies of class III; for $n = 6$, curves 1–3 correspond to bodies of class II and curves 4 and 5 to bodies of class III.

It follows from calculations that the use of optimum bodies allows a significant decrease in radiative heat flux to the body surface as compared to bodies of revolution with similar geometric characteristics (up to 50% for bodies of class III and up to 98% for bodies of class II). The greater the values of the radiation parameter m and the parameters S_*^2/V_* and V_*^2/C_{D_*}, the stronger the effect.

Note, however, the results obtained for star-shaped bodies of class II with a large number of cycles n should be treated cautiously, since the transverse contour of the body has deep troughs between two neighboring cycles, where the use of Eq. (1.1) for the radiative heat flux is physically unjustified. Therefore, it is necessary to further refine both the initial physical model for radiative heat transfer between bodies and troughs and the body shapes obtained by means of experimental studies and numerical calculations by more accurate models of spatial heat transfer.

The authors are grateful to A. I. Leont'ev and G. A. Tirskii for valuable comments.

This work was supported by the Russian Foundation for Fundamental Research (Grant Nos. 00-15-96030 and 01-01-00050) and by the Federal Goal-Oriented Program "Integration" [Grant No. I (A105-4)].

REFERENCES

1. É. Z. Apshtein, N. N. Pilyugin, and G. A. Tirskii, "Entrainment of mass and change in shape of a three-dimensional body moving along the trajectory in the Earth atmosphere," *Kosm. Issled.*, **17**, No. 2, 246–255 (1979).
2. M. A. Arguchintseva and N. N. Pilyugin, *Extreme Problems of Radiation Gas Dynamics* [in Russian], Izd. Mosk. Univ., Moscow (1997).
3. M. A. Arguchintseva and N. N. Pilyugin, "Extreme problems in heat transfer of spatial bodies moving with hypersonic velocities," *Prikl. Mat. Mekh.*, **56**, No. 4, 643–657 (1992).
4. M. A. Arguchintseva and N. N. Pilyugin, "Spatial shapes of bodies with minimum heating of the surface in hypersonic motion in the atmosphere," *Kosm. Issled.*, **30**, No. 5, 615–628 (1992).
5. N. N. Pilyugin and G. A. Tirskii, *Dynamics of an Ionized Radiating Gas* [in Russian], Izd. Mosk. Univ., Moscow (1989).
6. V. P. Stulov and E. G. Shapiro, "Radiation of the shock layer in a hypersonic air flow around blunted bodies," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 1, 154–160 (1970).
7. N. N. Pilyugin and G. A. Tirskii, "Hypersonic flow past of the plane blunt bodies by a nonviscous radiating gas," *J. Appl. Mech. Tech. Phys.*, **16**, No. 3, 365–370 (1975).
8. A. A. Deev, V. A. Levin, and N. N. Pilyugin, "Shape of a body with a minimum total flux of radiative energy to the body surface," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 4, 84–89 (1976).
9. É. Z. Apshtein and N. N. Pilyugin, "Rule of areas for the heat-transfer coefficient of spatial ablating bodies with heat fluxes depending locally on the surface slope," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 2, 71–76 (1979).
10. É. Z. Apshtein, N. V. Vartanyan, and V. I. Sakharov, "Determination of radiative heat fluxes on the surface of spatial and axisymmetric bodies in a supersonic ideal gas flow," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 1, 92–97 (1986).
11. É. Z. Apshtein, N. V. Vartanyan, and V. I. Sakharov, "Radiative heating of spatial bodies in a supersonic inviscid air flow," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 4, 183–187 (1986).
12. I. N. Murzinov, "Spherically blunted cones with minimum heat fluxes," *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, No. 6, 173–179 (1973).
13. J. H. Chin, "Shape change and conduction for nosetips at angle of attack," *AIAA J.*, **13**, No. 5, 599–604 (1975).
14. G. F. Widhoph, "Heat-transfer correlations for blunt cones at angle of attack," *J. Spacecraft Rockets*, **8**, No. 9, 1002–1004 (1971).
15. W. D. Hayes and R. F. Probstein, *Hypersonic Flow Theory*, Academic Press, New York (1959).
16. J. Kondo, J. Aihara, T. Tani, and A. Onij, "The optimum configuration and the optimum re-entry trajectory of space shuttle vehicles," ICAS Paper No. 72-27 (1972).
17. S. I. Anisimov, Ya. A. Imas, G. S. Romanov, and Yu. V. Khodyko, *Action of High-Power Radiation on Metals* [in Russian], Nauka, Moscow (1970).
18. A. Miele and J. Saaris, in: *Theory of Optimum Aerodynamic Shapes*, New York (1965).